**NASA TECHNICAL NOTE**

**NASA TN D-2416**

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NASA TN D-2416

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • AUGUST 1964

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For sale by the Office of Technical Services, Department of Commerce,
Washington, D.C. 20230 -- Price $1.00

# A SPRING-MASS REPRESENTATION OF A FREE-FREE NONUNIFORM BAR IN RESPONSE TO LONGITUDINAL FORCES

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## SUMMARY

A method is presented for solving the problem of the longitudinal response of a free-free nonuniform bar to an arbitrary force applied at one end, based upon a spring-mass model of the bar. The equations obtained are solved for the case of a uniform bar with a particular set of applied forces and the results are compared with the exact solutions of the corresponding boundary-value problem. The comparisons indicate that only a few mass points need be used in the spring-mass model to describe the displacements and a considerable number must be used to describe the forces accurately. Furthermore, it is indicated that the local accelerations are not readily obtainable by this method.

## INTRODUCTION

An exact solution of the boundary-value problem governing the response of a free-free nonuniform bar acted upon at one end by a time-dependent force is, in general, impossible or impractical to obtain. Therefore, either approximate methods of solution must be used or solutions for amenable problems of analogous systems or models. One analogous system with both conceptual simplicity and a problem that can be reduced to a matrix form easily programed for an automatic computer is the spring-mass system. With such a system the bar is represented by a finite number of mass points connected in series by linear, massless springs.

The purpose of this investigation is to gain some indication of what mass-point density is necessary to predict accurately displacements, stresses, and local accelerations. For this purpose the spring-mass model has been employed to derive the equations which approximately describe the displacements, stresses, and local accelerations at any point of the free-free nonuniform bar with an arbitrary force applied at one end. The equations obtained are solved for a 5- and a 20-spring-mass model of a free-free uniform bar with 4 different specific forces and the results are compared with the exact solution of the boundary-value problem which the spring-mass model attempts to represent. Smith in reference 1 used the spring-mass model in the study of the longitudinal response of a free-free bar which was impacted at one end, and Seigel and Waser in reference 2 used this model to determine the longitudinal response of a fixed-end bar subjected to a rectangular force pulse.

## SYMBOLS

| | |
|---|---|
| $A$ | cross-sectional area of bar |
| $a_n(\bar{x}) = 2n + 1 - \bar{x}$ | |
| $b_n(\bar{x}) = 2n + 1 + \bar{x}$ | |
| $E$ | Young's modulus of elasticity |
| $F$ | applied force |
| $\bar{F}(\bar{t}) = p(\bar{t})$ | parabolically rising forcing function |
| $\bar{F}(\bar{t}) = r(\bar{t})$ | ramp-rise forcing function |
| $\bar{F}(\bar{t}) = u(\bar{t})$ | unit-step forcing function |
| $\bar{F}(\bar{t}) = \delta(\bar{t})$ | impulsive forcing function |
| $\left[k\right]$ | square matrix defined by equation (24) |
| $k_n$ | spring constant of nth spring |
| $l$ | length of bar |
| $M$ | total mass of bar |
| $\lfloor m \rceil$ | diagonal matrix defined by equation (23) |
| $m_n$ | mass of nth mass point |
| $N$ | number of mass points in spring-mass model |
| $n$ | positive integer |
| $\bar{q}_n(\bar{t})$ | nth element in column matrix $\left\{\bar{q}(\bar{t})\right\}$ |
| $\left\{\bar{q}(\bar{t})\right\}$ | column matrix of functions to be determined |

$T = l\sqrt{\frac{\rho_r}{E_r}}$

| | |
|---|---|
| $t$ | time |
| $t_1, t_2$ | dimensional times (constants) |
| $x$ | distance measured along bar |
| $x_n$ | position of nth mass point, $\frac{l}{N}\left(n - \frac{1}{2}\right)$ |
| $Y_n(t)$ | displacement from initial position of nth mass point at time $t$ |
| $\left\{Y(t)\right\}$ | column matrix defined by equation (25) |
| $Y(x,t)$ | displacement of point x on bar at time $t$ |
| $\left[y\right]$ | modal matrix, $N$ by $N$ |
| $y_{ij}$ | jth element in eigenvector (mode) associated with ith eigenvalue |
| $\left\{y_n\right\}$ | nth eigenvector |
| $\rho$ | mass density |
| $\lfloor \omega^2 \rceil$ | spectral matrix, $N$ by $N$ |
| $\omega_n{}^2$ | nth eigenvalue associated with nth eigenvector |
| $\tau$ | variable of integration |

Subscripts:

| | |
|---|---|
| N | number of mass points in spring-mass model |
| n | positive integer |
| r | reference value |

Matrix notations:

| | |
|---|---|
| $\{\ \}$ | column matrix |

$[\tilde{\ }]$ transpose matrix

A dot over a symbol denotes the derivative with respect to time; two dots denote the second derivative with respect to time.

A bar over a symbol denotes a nondimensional quantity.

## ANALYSIS

### The Boundary-Value Problem

The longitudinal motion of a bar, initially at rest, with a time-dependent force applied at one end is determined by the following classic partial differential equation:

$$\frac{\partial}{\partial x}\left[E(x)\ A(x)\ \frac{\partial Y(x,t)}{\partial x}\right] = \rho(x)\ A(x)\ \frac{\partial^2 Y(x,t)}{\partial t^2} \tag{1}$$

and the conditions

$$\left.\begin{aligned} &Y(x,0) = 0 && E(x)\ A(x)\ \frac{\partial Y(x,t)}{\partial x}\bigg|_{x=0} = 0 \\ &\frac{\partial Y(x,t)}{\partial t}\bigg|_{t=0} = 0 && E(x)\ A(x)\ \frac{\partial Y(x,t)}{\partial x}\bigg|_{x=l} = F(t) \end{aligned}\right\} \tag{2}$$

where

| | |
|---|---|
| $t$ | time |
| $x$ | distance measured along bar |
| $l$ | length of bar |
| $Y(x,t)$ | displacement of point $x$ on bar at time $t$ |
| $E(x)$ | Young's modulus of elasticity for material of which bar is made |
| $\rho(x)$ | mass density of the bar material |

A(x) cross-sectional area of the bar

F(t) force applied at end of bar

For the nonuniform bar, equations (1) and (2) can be nondimensionalized by choosing reference values $E_r$, $\rho_r$, and $A_r$ and defining the following quantities:

$$\bar{x} = \frac{x}{l} \tag{3}$$

$$T = l\sqrt{\frac{\rho_r}{E_r}} \tag{4}$$

$$\bar{t} = \frac{t}{T} \tag{5}$$

$$\bar{E}(\bar{x}) = \frac{E(l\bar{x})}{E_r} \tag{6}$$

$$\bar{A}(\bar{x}) = \frac{A(l\bar{x})}{A_r} \tag{7}$$

$$\bar{\rho}(\bar{x}) = \frac{\rho(l\bar{x})}{\rho_r} \tag{8}$$

$$\bar{Y}(\bar{x},\bar{t}) = \frac{Y(l\bar{x},T\bar{t})}{l} \tag{9}$$

$$\bar{F}(\bar{t}) = \frac{1}{E_r A_r} F(T\bar{t}) \tag{10}$$

It follows that dimensionless forms of equations (1) and (2) can thus be expressed as

$$\frac{\partial}{\partial\bar{x}}\left[\bar{E}(\bar{x})\ \bar{A}(\bar{x})\ \frac{\partial\bar{Y}(\bar{x},\bar{t})}{\partial\bar{x}}\right] = \bar{\rho}(\bar{x})\ \bar{A}(\bar{x})\ \frac{\partial^2\bar{Y}(\bar{x},\bar{t})}{\partial\bar{t}^2} \tag{11}$$

and

$$\left.\begin{aligned} &\bar{Y}(\bar{x},0) = 0 \qquad && \left.\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}\right|_{\bar{x}=0} = 0 \\ &\left.\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{t}}\right|_{\bar{t}=0} = 0 \qquad && \bar{E}(\bar{x})\ \bar{A}(\bar{x}) \left.\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}\right|_{\bar{x}=1} = \bar{F}(\bar{t}) \end{aligned}\right\} \tag{12}$$

## Solutions for a Uniform Bar

For a uniform bar, equations (11) and (12) reduce to

$$\frac{\partial^2 \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}^2} = \frac{\partial^2 \bar{Y}(\bar{x},\bar{t})}{\partial \bar{t}^2} \tag{13}$$

and

$$\left.\begin{aligned} &\bar{Y}(\bar{x},0) = 0 \qquad && \left.\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}\right|_{\bar{x}=0} = 0 \\ &\left.\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{t}}\right|_{\bar{t}=0} = 0 \qquad && \left.\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}\right|_{\bar{x}=1} = \bar{F}(\bar{t}) \end{aligned}\right\} \tag{14}$$

The solution of these equations may be obtained by means of the Laplace transform. (See, for example, ch. 8 of ref. 3.) For present purposes the equations describing the displacement, force, and local acceleration at any section along the bar are desired. These parameters are given, respectively, by

$$\begin{aligned} \bar{Y}(\bar{x},\bar{t}) &= L^{-1}\left\{\frac{1}{s}\,\bar{f}(s)\,\frac{\cosh s\bar{x}}{\sinh s}\right\} \\ &= L^{-1}\left\{\frac{1}{s}\,\bar{f}(s)\sum_{n=0}^{\infty}\left[e^{-a_n(\bar{x})s} + e^{-b_n(\bar{x})s}\right]\right\} \\ &= \sum_{n=0}^{\infty}\int_0^{\bar{t}} \bar{F}(\bar{t}-\tau)\left\{H\left[\tau-a_n(\bar{x})\right] + H\left[\tau-b_n(\bar{x})\right]\right\} d\tau \end{aligned} \tag{15}$$

$$\frac{\partial\overline{Y}(\overline{x},\overline{t})}{\partial\overline{x}} = L^{-1}\left\{\overline{f}(s)\ \frac{\sinh\ s\overline{x}}{\sinh\ s}\right\}$$

$$= \sum_{n=0}^{\infty}\left\{\overline{F}\left[\overline{t}-a_n(\overline{x})\right]\ H\left[\overline{t}-a_n(\overline{x})\right] - \overline{F}\left[\overline{t}-b_n(\overline{x})\right]\ H\left[\overline{t}-b_n(\overline{x})\right]\right\} \tag{16}$$

and

$$\frac{\partial^2 Y(\overline{x},\overline{t})}{\partial\overline{t}^2} = L^{-1}\left\{s\ \overline{f}(s)\ \frac{\cosh\ s\overline{x}}{\sinh\ s}\right\}$$

$$= \sum_{n=0}^{\infty}\left\{\frac{\partial\overline{F}\left[\overline{t}-a_n(\overline{x})\right]}{\partial\overline{t}}\ H\left[\overline{t}-a_n(\overline{x})\right] + \frac{\partial\overline{F}\left[\overline{t}-b_n(\overline{x})\right]}{\partial\overline{t}}\ H\left[\overline{t}-b_n(\overline{x})\right]\right\} \tag{17}$$

where

$a_n(\overline{x}) = 2n + 1 - \overline{x}$

$b_n(\overline{x}) = 2n + 1 + \overline{x}$

$\overline{f}(s)$ is the inverse Laplace transform of $\overline{F}(\overline{t})$ or, symbolically,

$L^{-1}\left\{\overline{f}(s)\right\} = \overline{F}(\overline{t})$, where $s$ is a parameter

$H\left[\ \right]$ is a function which equals 1 if the term in brackets is positive and equals 0 if the term in brackets is negative

Equations (15), (16), and (17) state mathematically the fact that the signal is transmitted along the bar at a velocity equal to $\sqrt{\frac{E_r}{\rho_r}}$ and that when a signal arrives at either end of the bar it is reflected. Thus, it may be noted that the constant $T$ given by equation (4) is the time required for a signal to traverse the length of the bar.

Equations (15), (16), and (17) represent an exact solution to the boundary-value problem stated and will be evaluated and discussed subsequently for four particular forcing functions $\overline{F}(\overline{t})$. It should be mentioned that the acceleration (eq. (17)) is directly proportional to the derivative of the forcing function with respect to time, with a time delay; consequently, some care must be taken in trying to assign physical meaning to local acceleration in which $\bar{F}(\bar{t})$ or $\frac{\partial \bar{F}(\bar{t})}{\partial \bar{t}}$ is discontinuous.

## Differential Equations for the Spring-Mass Representation of a Bar

As indicated in the introduction, the boundary-value problem for a nonuniform bar, given by equations (1) and (2), is solvable only in a few very special cases. However, the equations describing the motion of a spring-mass model of this bar are more tractable and may be solved by matrix techniques. The governing equations along with a systematic method of solution are derived in the following paragraphs.

Figure 1 shows a schematic representation of the spring-mass model of a nonuniform bar. In general, the location and mass of the mass points is a matter of engineering judgment. In some applications it is convenient to take equal intervals between successive mass points and to assign to the nth mass point at $x_n$ a mass represented by

$$m_n = \int_{\frac{x_{n-1}+x_n}{2}}^{\frac{x_n+x_{n+1}}{2}} m(x)\ dx \tag{18a}$$

where $m(x)$ is the mass per unit length of the bar. For the uniform bar, all the mass stations may be evenly spaced and equation (18a) reduces to

$$m_n = \frac{M}{N} \tag{18b}$$

for all $n$ where $M$ is the total mass of the bar and $N$ is the total number of mass points.

For either a uniform or a nonuniform bar, the spring constant $k_n$ of the spring connecting the mass points $m_n$ and $m_{n+1}$ is given by

$$\frac{1}{k_n} = \int_{x_n}^{x_{n+1}} \frac{dx}{E(x)\ A(x)} \tag{19a}$$

which for the uniform bar reduces to

$$k_n = \frac{NEA}{l} \tag{19b}$$

Let $Y_n(t)$ be the displacement (positive up in fig. 1) at time $t$ of the nth mass point from its initial position. Then the response of the system which is initially at rest and is excited at the Nth mass point by a force $F(t)$ is described by a system of equations which takes the matrix form

$$\lfloor m \rceil \{\ddot{Y}(t)\} + [k]\{Y(t)\} = \{F(t)\} \tag{20}$$

and the initial conditions

$$\{Y(0)\} = \{0\} \tag{21}$$

$$\{\dot{Y}(0)\} = \{0\} \tag{22}$$

where

$$\lfloor m \rceil = \begin{bmatrix} m_1 & & & & & 0 \\ & m_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & m_N \end{bmatrix} \tag{23}$$

$$[k] = \begin{bmatrix} k_1 & -k_1 & & & & & 0 \\ -k_1 & k_1 + k_2 & -k_2 & & & & \\ & -k_2 & k_2 + k_3 & -k_3 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & -k_{N-1} & k_{N-1} + k_N & -k_N \\ 0 & & & & & -k_N & k_N \end{bmatrix} \tag{24}$$

$$\left\{Y(t)\right\} = \begin{Bmatrix} Y_1(t) \\ Y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ Y_N(t) \end{Bmatrix} \tag{25}$$

and

$$\left\{F(t)\right\} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ F(t) \end{Bmatrix} \tag{26}$$

By means of equations (4), (5), and (10) and the following four equations

$$\left\{\overline{Y}(\overline{t})\right\} = \frac{1}{l}\left\{Y(t)\right\} \tag{27}$$

$$\lfloor \overline{m} \rceil = \frac{1}{M}\lfloor m \rceil \tag{28}$$

$$\left\{\overline{F}(\overline{t})\right\} = \frac{1}{E_r A_r}\left\{F(T\overline{t})\right\} \tag{29}$$

$$\left[\overline{k}\right] = \left[k\right] \int_0^l \frac{dx}{E(x)\ A(x)} \tag{30}$$

equations (20), (21), and (22) can be written in dimensionless form as follows:

$$\left[\bar{m}\right]\left\{\ddot{\bar{Y}}(\bar{t})\right\} + \left[\bar{k}\right]\left\{\bar{Y}(\bar{t})\right\} = \left\{\bar{F}(\bar{t})\right\} \tag{31}$$

$$\left\{\bar{Y}(0)\right\} = \left\{0\right\} \tag{32}$$

$$\left\{\dot{\bar{Y}}(0)\right\} = \left\{0\right\} \tag{33}$$

## Solution of the Differential Equations

The classical method of solving equation (31) is to expand the unknown $\bar{Y}(\bar{t})$ in terms of the normal modes of oscillation of the homogeneous equation

$$\left[\bar{m}\right]\left\{\ddot{\bar{Y}}(\bar{t})\right\} + \left[\bar{k}\right]\left\{\bar{Y}(\bar{t})\right\} = \left\{0\right\} \tag{34}$$

(See ch. 1 of ref. 4.) This leads to a system of uncoupled ordinary differential equations. For simple harmonic motion this equation becomes

$$\left[\bar{k}\right]\left\{y_n\right\} = \omega_n^2\left[\bar{m}\right]\left\{y_n\right\} \tag{35}$$

The $N$ orthogonal eigenvectors $\left\{y_n\right\}$ and their $N$ associated eigenvalues $\omega_n^2$ of equation (35), which can be rapidly and accurately obtained by the Serial-Jacobi method[1], are the $N$ normal modes and frequencies of equation (34). The modal matrix $\left[y\right]$ of the system is defined to be an $N$ by $N$ matrix whose columns are the normal modes $\left\{y_n\right\}$. The normal modes of equation (34) are determined only to within a multiplicative constant. In the following work it is convenient to choose this constant so that $\left[y\right]$ satisfies the condition of orthonormality which is

$$\left[\tilde{y}\right]\left[\bar{m}\right]\left[y\right] = \left[1\right] \tag{36}$$

where $\left[\tilde{y}\right]$ is the transpose of $\left[y\right]$.

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[1]Individuals having access to data processing installations holding membership in SHARE may refer to program NY EVV.

An $N$ by $N$ diagonal matrix $\lfloor \omega^2 \rceil$ which will also be used is constructed so that the element on the diagonal in the ith column is the eigenvalue associated with the eigenvector in the ith column of $\left[ y \right]$. Substituting

$$\left\{ \overline{Y}(\overline{t}) \right\} = \left[ y \right] \left\{ \overline{q}(\overline{t}) \right\} \tag{37}$$

into equations (31), (32), and (33) and premultiplying each term in the resulting expressions by the transpose of the modal matrix $\left[ y \right]$ leads to a system of uncoupled equations, namely (see, for example, ref. 5)

$$\left\{ \ddot{\overline{q}}(\overline{t}) \right\} + \lfloor \omega^2 \rceil \left\{ \overline{q}(\overline{t}) \right\} = \left[ \tilde{y} \right] \left\{ \overline{F}(\overline{t}) \right\} \tag{38}$$

with initial conditions as follows:

$$\left\{ \overline{q}(0) \right\} = \left\{ 0 \right\} \tag{39}$$

$$\left\{ \dot{\overline{q}}(0) \right\} = \left\{ 0 \right\} \tag{40}$$

From the $N$ uncoupled ordinary differential equations in equation (38) and the initial conditions given in equations (39) and (40) it is possible to determine the following formula for the general element $\overline{q}_n(\overline{t})$ in the column matrix $\left\{ \overline{q}(\overline{t}) \right\}$:

$$\overline{q}_n(\overline{t}) = y_{nN} \int_0^{\overline{t}} \frac{\sin \omega_n \tau}{\omega_n} \overline{F}(\overline{t} - \tau)\, d\tau \tag{41}$$

where $y_{nN}$ is the Nth element in the nth eigenvector and $\omega_n$ is the natural frequency associated with the nth eigenvector. Then, by means of the matrix equation (37), it is possible to determine the displacement $\overline{Y}_n(\overline{t})$ of each mass point. The local acceleration $\ddot{\overline{Y}}_n(\overline{t})$ of the mass point at station $\overline{x}_n$ can be found by differentiating matrix equation (37) - that is

$$\left\{ \ddot{\overline{Y}}(\overline{t}) \right\} = \left[ y \right] \left\{ \ddot{\overline{q}}(\overline{t}) \right\} \tag{42}$$

where $\ddot{\overline{q}}_n(\overline{t})$ can be found from equation (38). The total longitudinal force at any point $\overline{x}$ on the spring connecting the n and (n - 1) mass of a uniform bar is taken to be

$$\frac{\partial\overline{Y}(\overline{x},\overline{t})}{\partial\overline{x}} \approx N\left[\overline{Y}_n(\overline{t}) - \overline{Y}_{n-1}(\overline{t})\right] \tag{43}$$

for all $\overline{x}$ in the internal $\overline{x}_{n-1} \leqq \overline{x} \leqq \overline{x}_n$.

## SPECIFIC FORCING FUNCTIONS

The specific dimensionless forcing functions $\overline{F}(\overline{t})$ used in the calculations presented in this paper are the following:

(1) The impulsive forcing function $\overline{F}(\overline{t}) = \delta(\overline{t})$ where

$$\int_{-\infty}^{\infty} \delta(\overline{t})\ d\overline{t} = 1 \tag{44}$$

and

$$\delta(\overline{t}) = 0 \qquad (\overline{t} \neq 0)$$

(2) The unit-step forcing function $\overline{F}(\overline{t}) = u(\overline{t})$ where

$$\left.\begin{aligned} u(\overline{t}) &= 0 \qquad & (\overline{t} < 0) \\ u(\overline{t}) &= 1 \qquad & (\overline{t} > 0) \end{aligned}\right\} \tag{45}$$

(3) The ramp-rise forcing function $\overline{F}(\overline{t}) = r(\overline{t})$ where

$$\left.\begin{aligned} r(\overline{t}) &= 0 \qquad & (\overline{t} < 0) \\ r(\overline{t}) &= \frac{\overline{t}}{\overline{t}_2} \qquad & \left(0 \leqq \overline{t} \leqq \overline{t}_2\right) \\ r(\overline{t}) &= 1 \qquad & \left(\overline{t}_2 \leqq \overline{t}\right) \end{aligned}\right\} \tag{46}$$

(4) The parabolically rising forcing function $\bar{F}(\bar{t}) = p(\bar{t})$ where

$$\left.\begin{aligned}
p(\bar{t}) &= 0 && (\bar{t} \leqq 0) \\
p(\bar{t}) &= \frac{\bar{t}^2}{\bar{t}_1\bar{t}_2} && \left(0 \leqq \bar{t} \leqq \bar{t}_1\right) \\
p(\bar{t}) &= \frac{\bar{t}_1}{\bar{t}_1 - \bar{t}_2} - \frac{2\bar{t}}{\bar{t}_1 - \bar{t}_2} + \frac{\bar{t}^2}{\bar{t}_2\left(\bar{t}_1 - \bar{t}_2\right)} && \left(\bar{t}_1 \leqq \bar{t} \leqq \bar{t}_2\right) \\
p(\bar{t}) &= 1 && \left(\bar{t}_2 \leqq \bar{t}\right)
\end{aligned}\right\} \tag{47}$$

Graphs of the three forces described in equations (45) to (47) are shown plotted as functions of time in figure 2.

The values of $\bar{t}_1$ and $\bar{t}_2$ used in the calculations were obtained by taking $T = 0.352$ sec, $t_1 = 0.05$ sec, and $t_2 = 0.1$ sec which by equation (5) implies that $\bar{t}_1 = 0.142$ and $\bar{t}_2 = 0.284$. The value of $T$ chosen is representative of the time required for a signal to traverse the length of a large Nova class vehicle, and the quantity $t_2$ is representative of the time required for a solid-fuel rocket engine to achieve maximum thrust. The last three forcing functions are chosen to be representative of the longitudinal forces experienced by missile-type structures during lift-off. They were chosen so that a study could be made of the effect of the time history of the applied force prior to attaining its maximum on the forces and local accelerations in the bar.

## RESULTS AND DISCUSSION

The equations describing the displacements, forces, and local accelerations of points along a uniform bar, obtained approximately by the analysis just presented and given exactly by equations (15), (16), and (17), have been evaluated by using the four forcing functions defined and a 5- and a 20-spring-mass representation of the bar. By comparing the results, some indication can be obtained of how well this spring-mass solution approaches the exact solution as the mass-point density is increased.

The comparisons made between the solution of the problem of the spring-mass representation of a uniform bar and the exact solution of the corresponding boundary-value problem are shown in figures 3 to 15. Each of the figures shows either dimensionless displacements, forces, or local accelerations plotted as functions of dimensionless time. The solid-line curve in each figure represents results from the exact solution of the boundary-value problem, and the dashed-line curve represents results from approximate solution. For convenience in making comparisons, figures 3 to 15 have been arranged in three sets so that figures 3 to 5 are those associated with displacements, figures 6 to 11 are those associated with forces, and figures 12 to 15 are those associated with local accelerations.

In the set of figures pertaining to displacements, the spring-mass solutions obtained at specific mass stations $\overline{x}_n$ are compared with the exact solutions obtained for $\overline{x} = \overline{x}_n$ on the uniform bar. In figures 3 and 4, the displacement history of specific points along the bar excited by an impulsive force is shown. The exact solution shows that the displacement curve increases in a series of steps. In figure 3 it is seen that the 5-spring-mass solution gives a fair representation of the displacement. However, as should be expected, the 20-spring-mass solution shown in figure 4 comes closer to representing the details of the exact solution in some respects.

In figure 5 the displacement, calculated exactly, due to a unit force is shown and is compared with a 5-spring-mass solution. The curves based on this solution are a very close approximation to those based on the exact solution. It should be noted that the displacement shown is predominantly rigid-body motion. The 20-spring mass solution will naturally approximate the exact solution with a greater degree of accuracy than the 5-spring-mass solution but the difference between these two solutions is very slight and hence the 20-spring-mass solution is not shown.

Furthermore, the displacements associated with the ramp rise and the parabolically rising force are also omitted since they are predominantly rigid-body motion and are almost the same as the displacements obtained with a unit forcing function.

Figures 6 to 11 pertain to the forces. In these figures the approximate force obtained in the interval $\overline{x}_{n-1} \leqq \overline{x} \leqq \overline{x}_n$ from equation (43) is compared with the exact solution determined at $\overline{x} = \frac{1}{2}\left[\overline{x}_{n-1} + \overline{x}_n\right]$. It should be recalled that the approximate force is constant over the interval because of an absence of inertial forces within the interval. Figures 6 and 7 are associated with the unit forcing function, figures 8 and 9, with a ramp-rise force, and figures 10 and 11, with a parabolically rising force. The 5-spring-mass solutions are superimposed on the exact solution in figures 6, 8, and 10, and the 20-spring-mass solutions are superimposed in figures 7, 9, and 11. It is evident from figures 6 to 11 that the curves obtained from the 20-spring-mass solution can approximate the curve obtained from the boundary-value problem with a higher degree of accuracy than can those based on the 5-spring-mass solution. Furthermore, it can be observed in figures 6, 8, and 10 that the curves describing the forces existing in the springs of the 5-spring-mass model are nearly independent of the applied forces considered; that is, the shape and amplitude of the curves in figures 6, 8, and 10 are essentially the same. In figures 7, 9, and 11 the curves describing the forces in the springs of the 20-spring-mass model would follow the curves obtained from the exact solution of the boundary-value problem almost exactly if it were not for the oscillations which are superimposed on the general shape of the approximate curve. The amplitude of these oscillations about the exact curve increases as the peak values of the first derivatives of the applied forces with respect to time increase.

The exact and approximate curves describing the time history of the local accelerations are shown in figures 12 to 15. In these figures the results of the approximate solutions obtained at the specific mass stations $\bar{x}_n$ are compared with the results of the exact solution obtained for $\bar{x} = \bar{x}_n$. Only the accelerations associated with the ramp-rise force (figs. 12 and 13) and the parabolically rising force (figs. 14 and 15) are shown since these two forces are the only ones considered which lead to meaningful results. It may be recalled that only the ramp-rise and parabolically rising forces lead to acceleration histories which are realizable. In figures 12 and 14 the results of the 5-spring-mass solution are shown, and in figures 13 and 15 the results of the 20-spring-mass solution are shown.

In figures 12 and 14, associated with the 5-spring-mass solution, it may be seen that the curves describing the local acceleration of the mass points associated with either the ramp-rise or the parabolically rising force have the same general shape but differ in magnitude. It may also be noted that, in both cases, all the peak values obtained from the exact solution of the boundary-value problem are grossly underestimated by the 5-spring-mass solution. By examination of equation (17) and figures 12 to 15 it may be found that the local acceleration obtained from the exact solution to the boundary-value problem is made up of a series of pulses of duration $\bar{t}_2 = t_2/T$ where $t_2$ is the time required from $t = 0$ for the force to reach its constant value. In addition, it should be noted that near either end of the bar these pulses overlap and reinforce each other; these pulses will hereinafter be referred to as reinforced pulses. For example, reinforced rectangular pulses are seen to exist at stations 0.1 and 0.9 in figure 12 and reinforced triangular pulses are seen to exist at stations 0.1 and 0.9 in figure 14.

The 20-spring-mass solutions of the local accelerations are shown in figures 13 and 15. In figure 13, which is associated with the ramp-rise force it may be seen that the 20-spring-mass solution is capable of predicting the arrival time and the magnitude of the rectangular pulses which are not reinforced. For the reinforced pulses it becomes evident that if the reinforced part is of very short duration the peak values of the reinforced pulse will be underestimated. In figure 15, which is associated with the parabolically rising force, the curve for the 20-spring-mass solution tends to round off and hence to underpredict the peak values of all the triangular pulses which are not reinforced, that is, those at stations 0.575 and 0.175. For the stations at which reinforced triangular pulses occur, the approximate solution again underestimates the peak values at all stations shown except at station 0.075. It should be observed, however, that at station 0.075, there is a relatively long time between the two peaks of the reinforced pulse, for which the acceleration is constant and that this constant is very nearly equal to the peak value of the acceleration at station 0.075.

The existence of oscillations with large positive and negative amplitude in the 20-spring-mass solution is more pronounced in the curves which attempt to predict the local accelerations than it is in the curves which attempt to predict the forces at points along the uniform bar. These oscillations are inherent in the spring-mass model and tend to mask the exact solution of the boundary-value problem. Thus, it is impossible to predict accurately the peak values of the local acceleration at all points along the bar with a spring-mass model having 20 or fewer mass points.

## CONCLUDING REMARKS

In the present paper an attempt has been made to evaluate an approximate method of solving the problem of the longitudinal response of a nonuniform bar with an arbitrary force applied at one end. The method involved replacing the bar by a spring-mass model. The equations describing the displacements, forces, and local accelerations at points in this model were obtained by means of a modal analysis. These equations were specialized to the case of a uniform bar, for which the problem of longitudinal responses can be solved exactly. Comparisons were then made between the exact and the approximate expressions describing the displacements, forces, and local accelerations at specific points along the uniform bar for specific forces. For the forces which were considered, the displacement history of any point along the uniform bar was found to be predominantly rigid-body motion and, therefore, independent of the mass-point density. It also appears that, for these applied forces, the force history at any point along the uniform bar could be approximated satisfactorily with 20 mass points and, in addition, it was shown that a 20-spring-mass model is not capable of accurately predicting the actual acceleration history at all the points along the bar.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., April 11, 1964.

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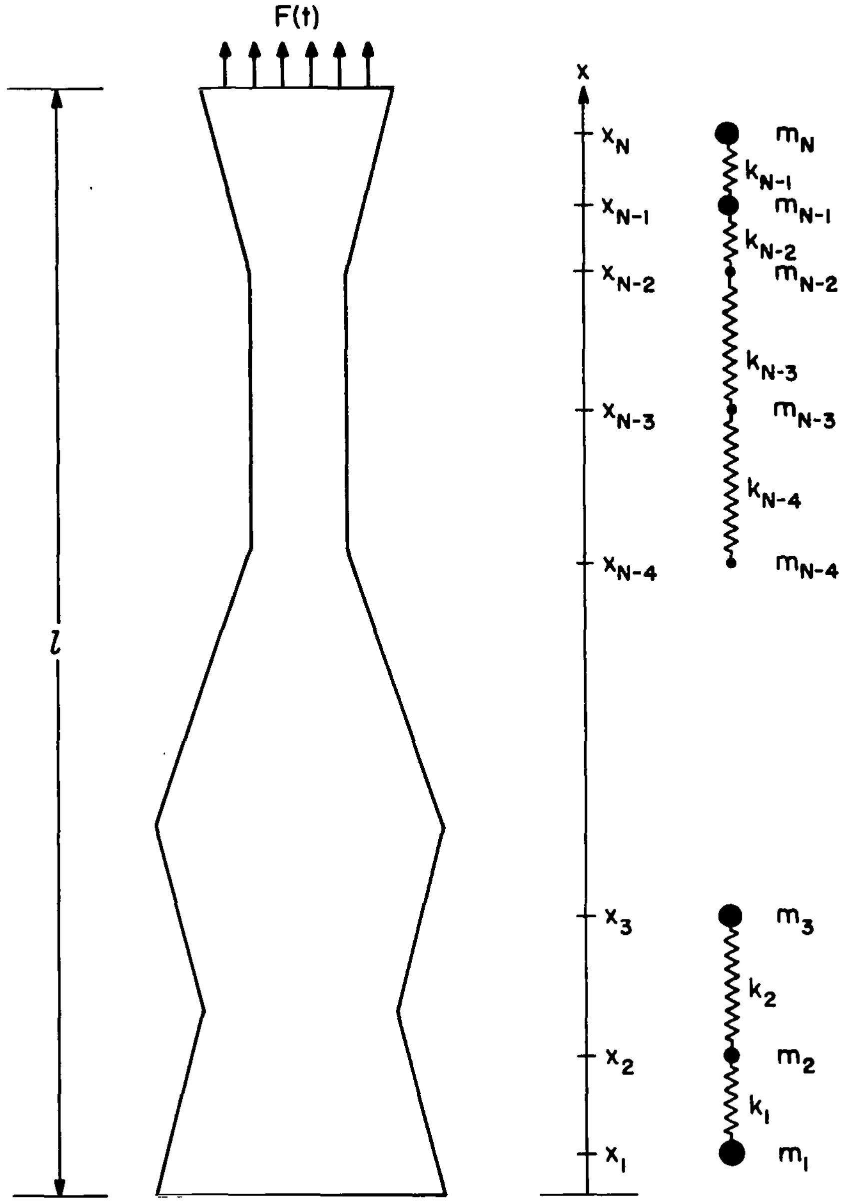
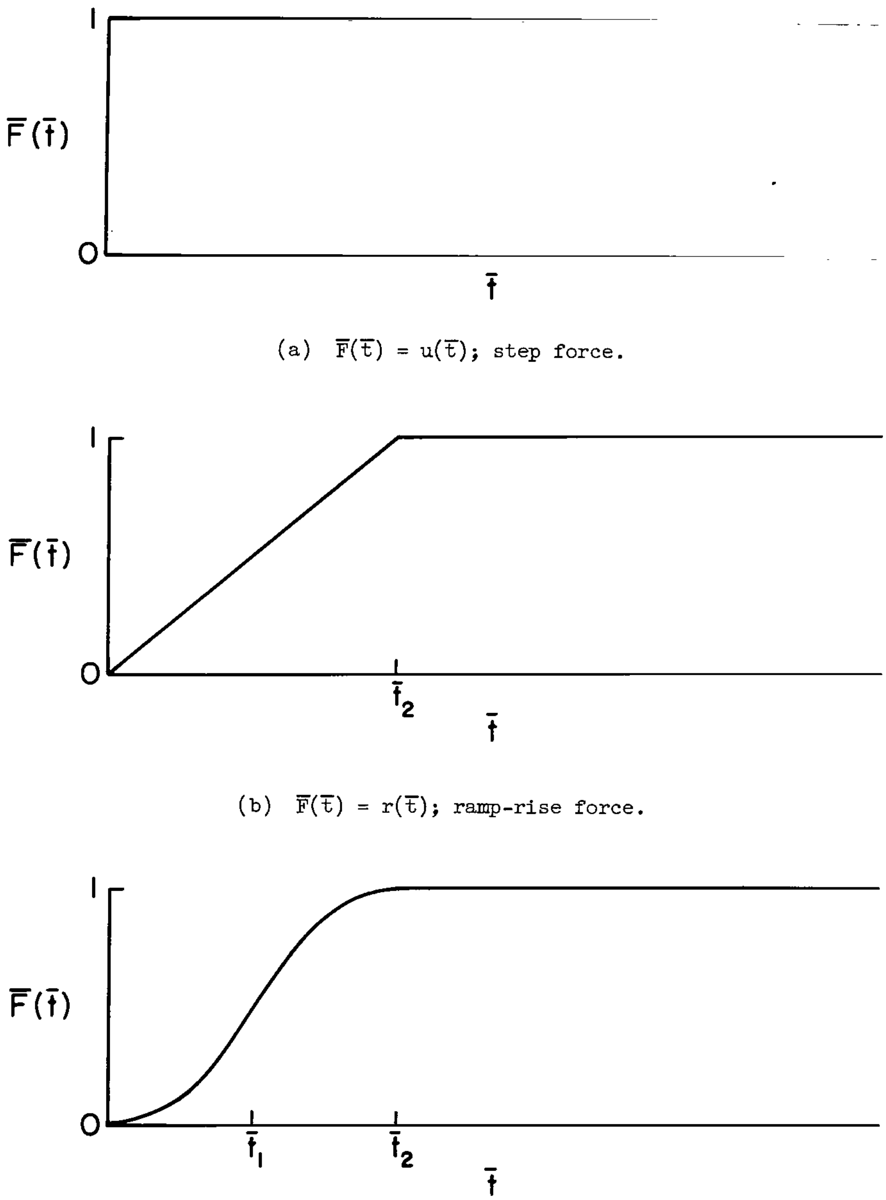



(a) Nonuniform bar. (b) Spring-mass analogy.

Figure 1.- Nonuniform bar and its spring-mass analogy with applied force F(t) applied at one end.




(a) $\bar{F}(\bar{t}) = u(\bar{t})$; step force.



(b) $\bar{F}(\bar{t}) = r(\bar{t})$; ramp-rise force.



(c) $\bar{F}(\bar{t}) = p(\bar{t})$; parabolically rising force.

Figure 2.- Sketch of different applied forces acting at station one.

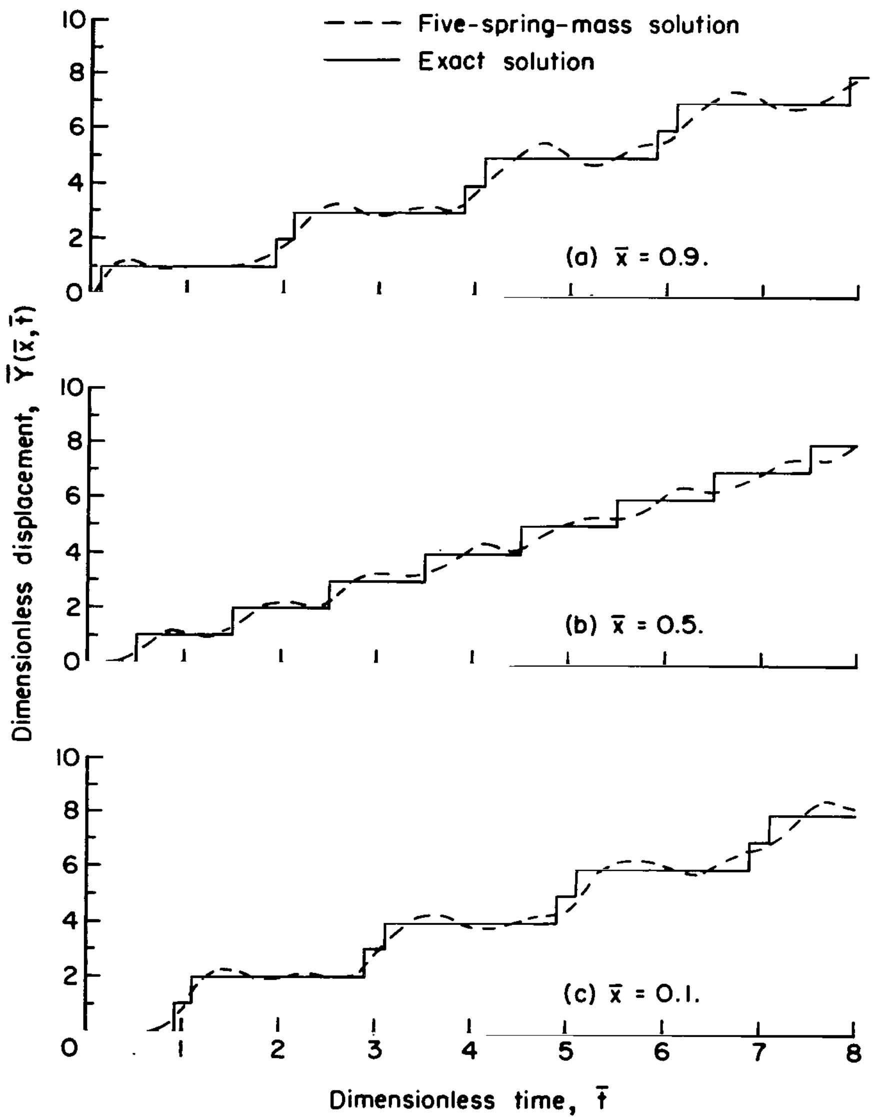



Figure 3.- Comparison between displacement $\bar{Y}(\bar{x},\bar{t})$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = \delta(\bar{t})$.

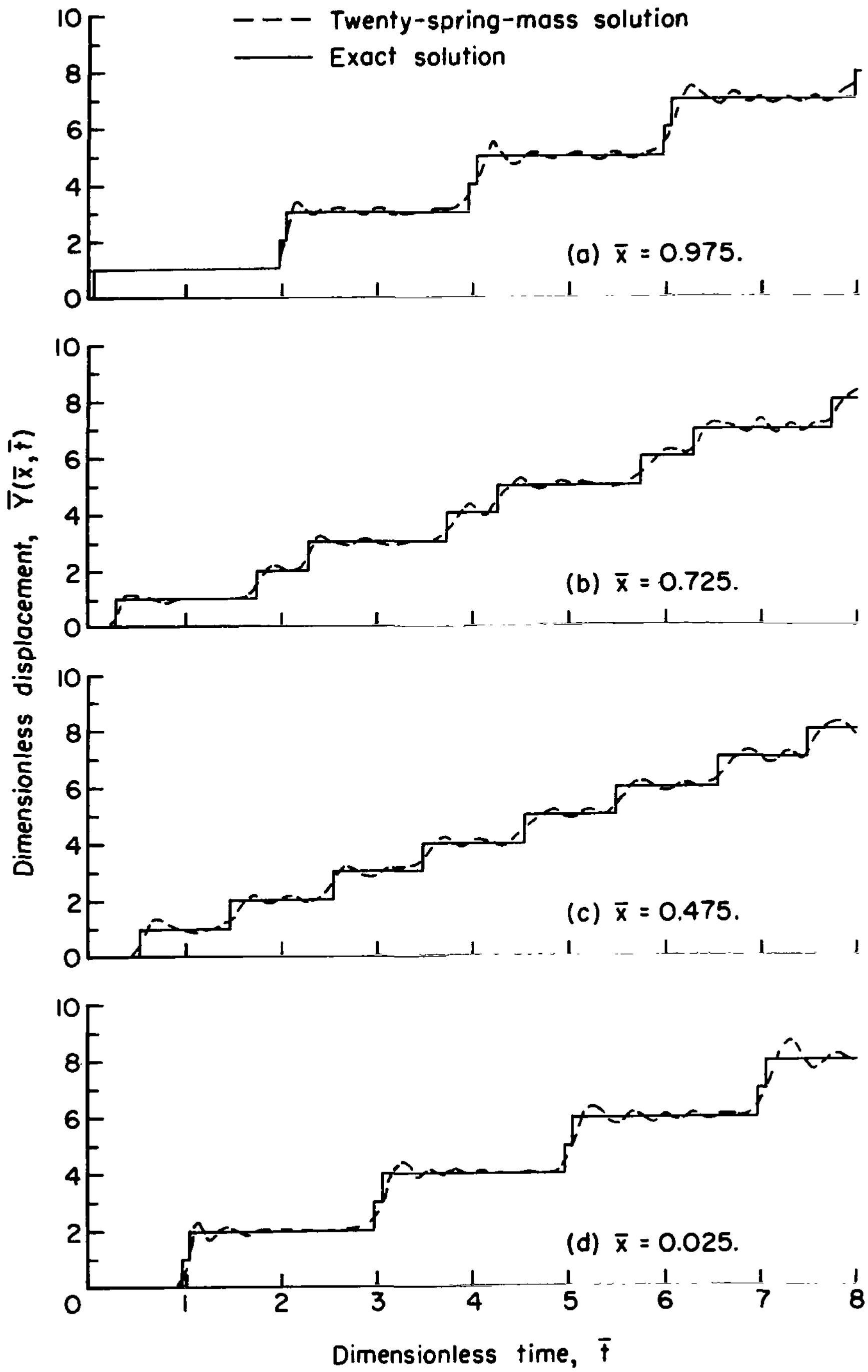



Figure 4.- Comparison between displacement $\bar{Y}(\bar{x},\bar{t})$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = \delta(\bar{t})$.

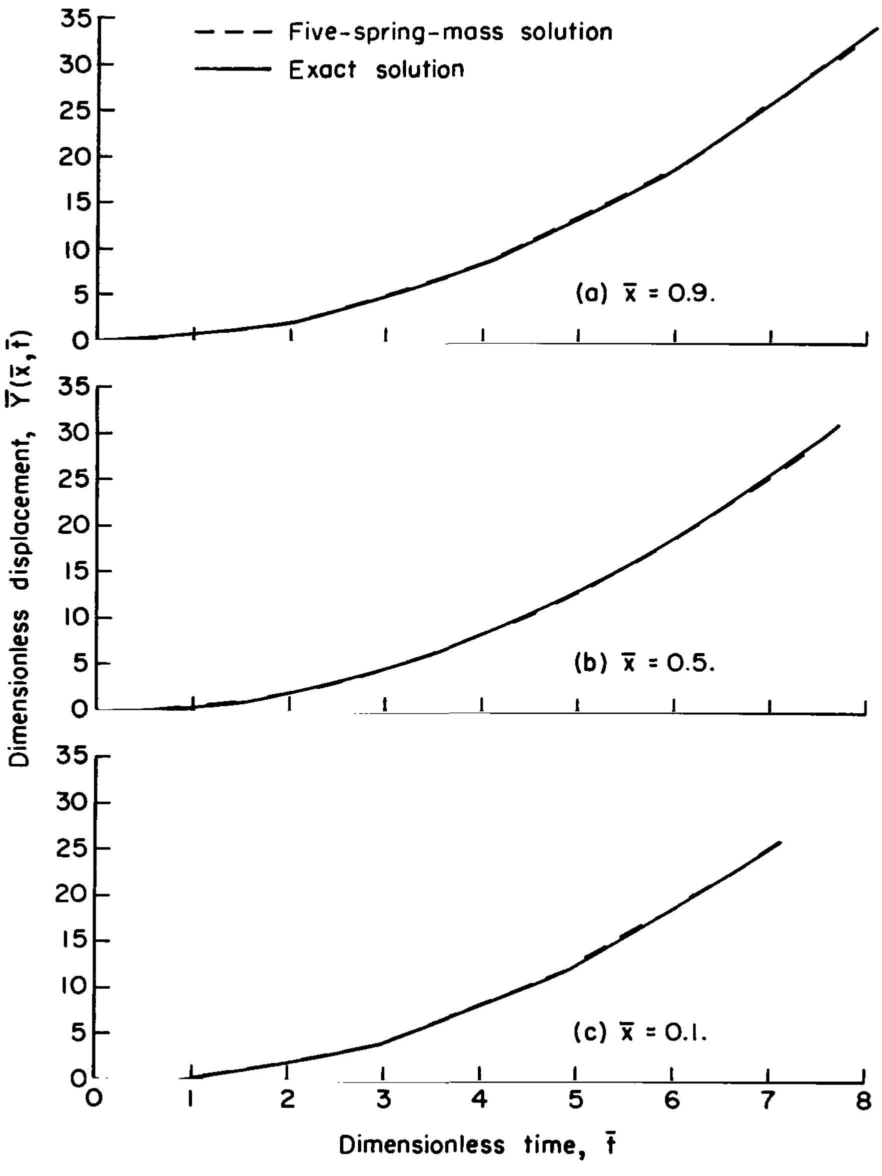



Figure 5.- Comparison between displacement $\bar{Y}(\bar{x},\bar{t})$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = u(\bar{t})$. (See fig. 2 for sketch of $u(\bar{t})$.)

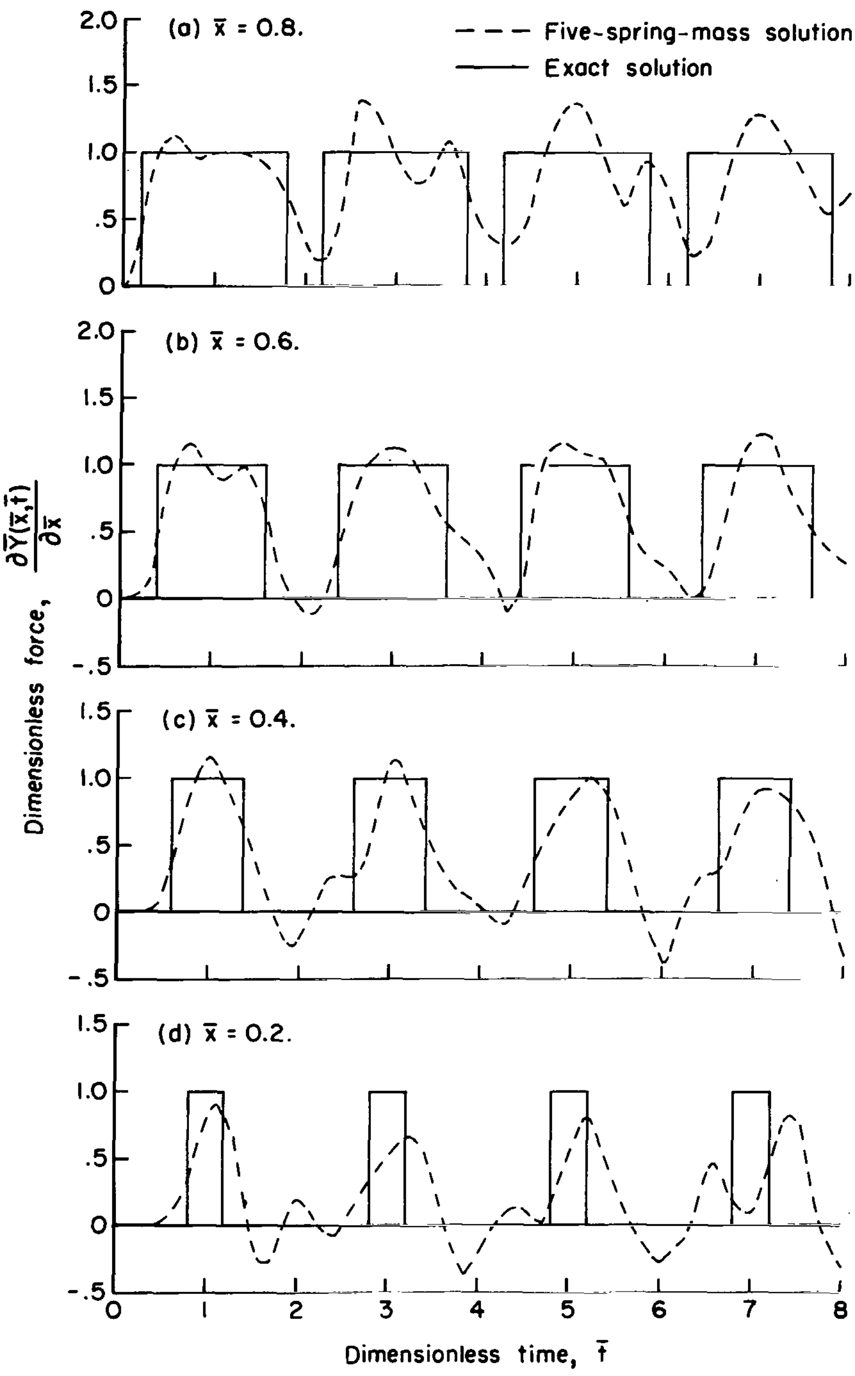



Figure 6.- Comparison between force $\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = u(\bar{t})$. (See fig. 2 for sketch of $u(\bar{t})$.)

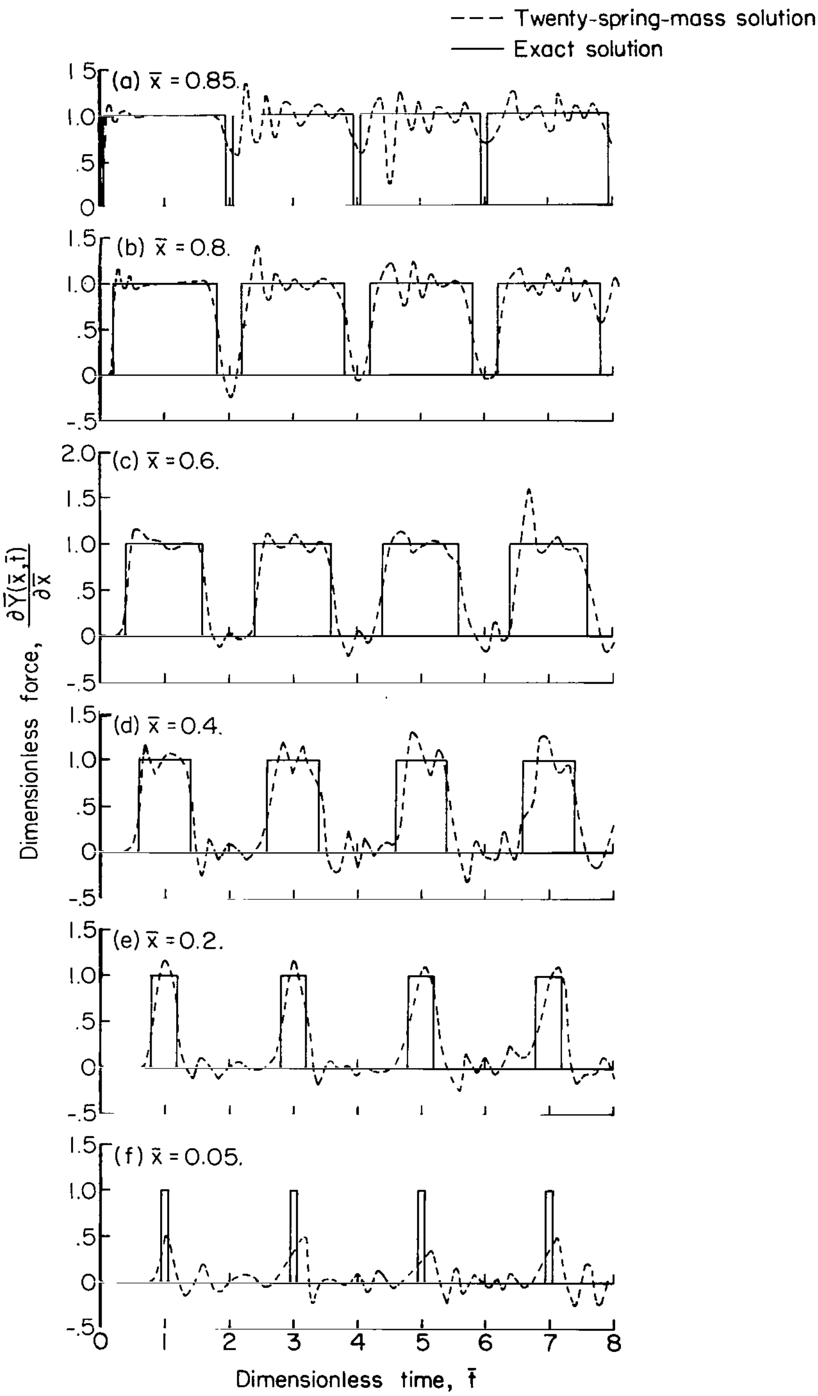



Figure 7.- Comparison between force $\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = u(\bar{t})$. (See fig. 2 for sketch of $u(\bar{t})$.)

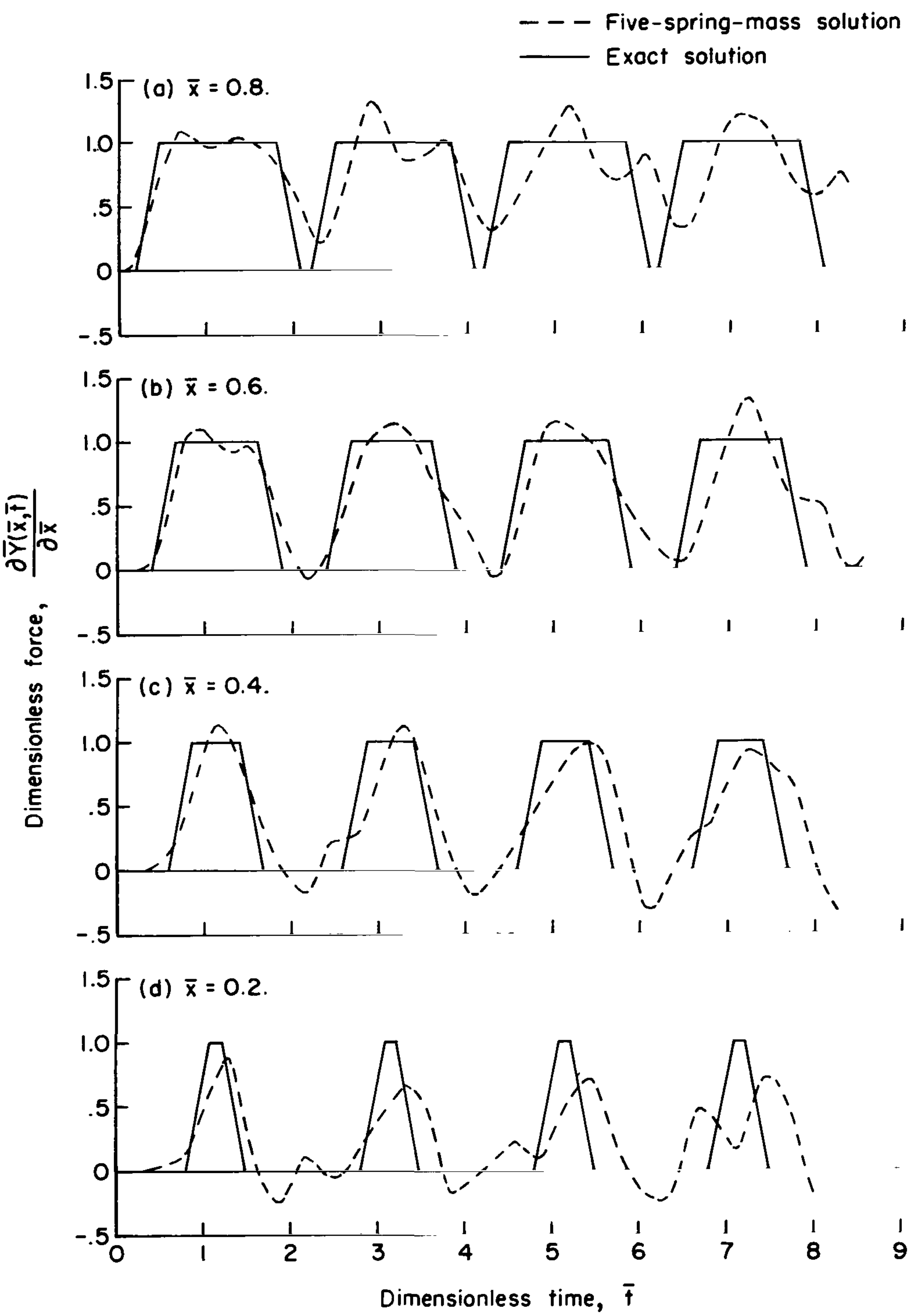



Figure 8.- Comparison between force $\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = r(\bar{t})$. (See fig. 2 for sketch of $r(\bar{t})$.)

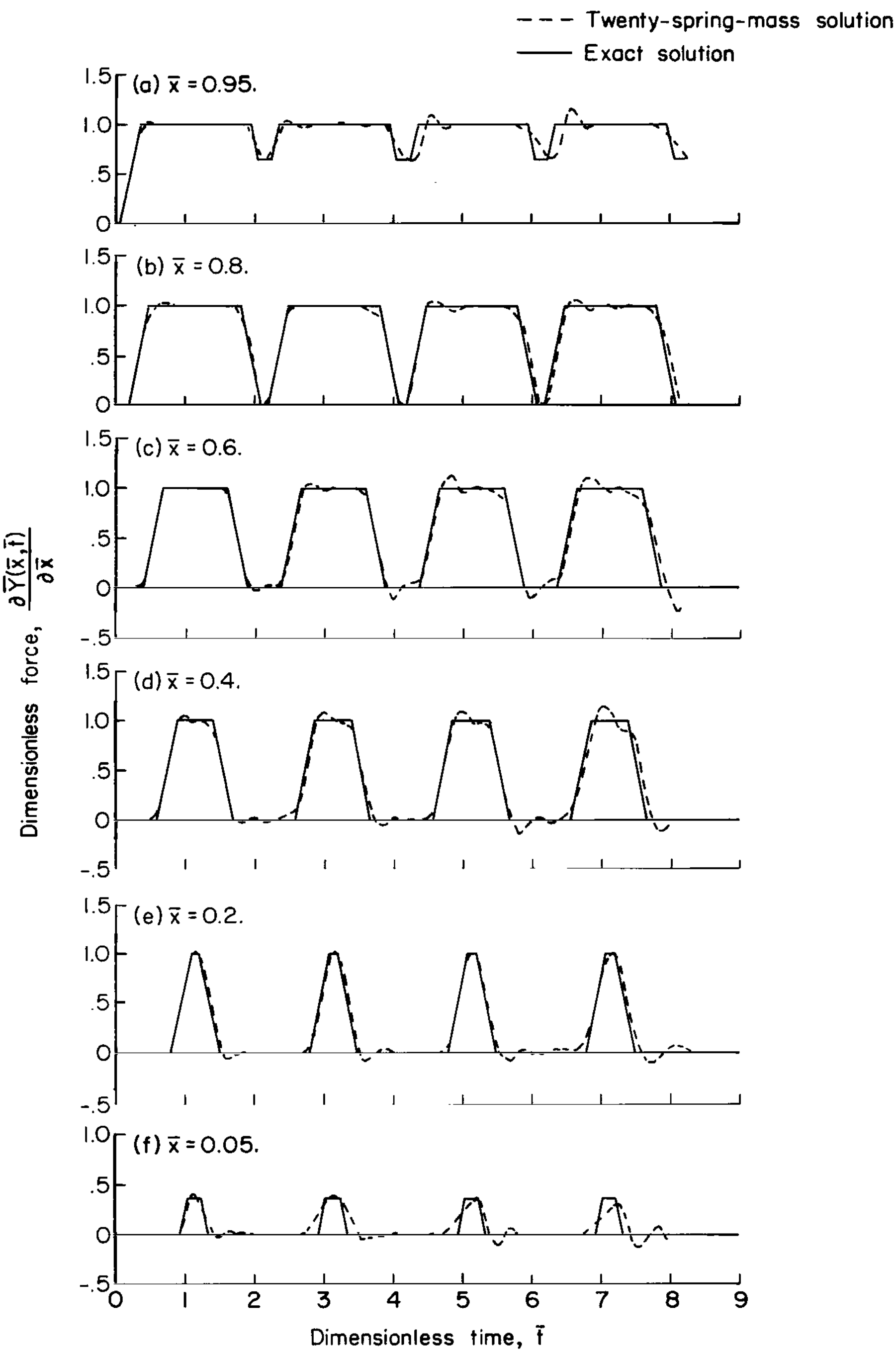



Figure 9.- Comparison between force $\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = r(\bar{t})$. (See fig. 2 for sketch of $r(\bar{t})$.)

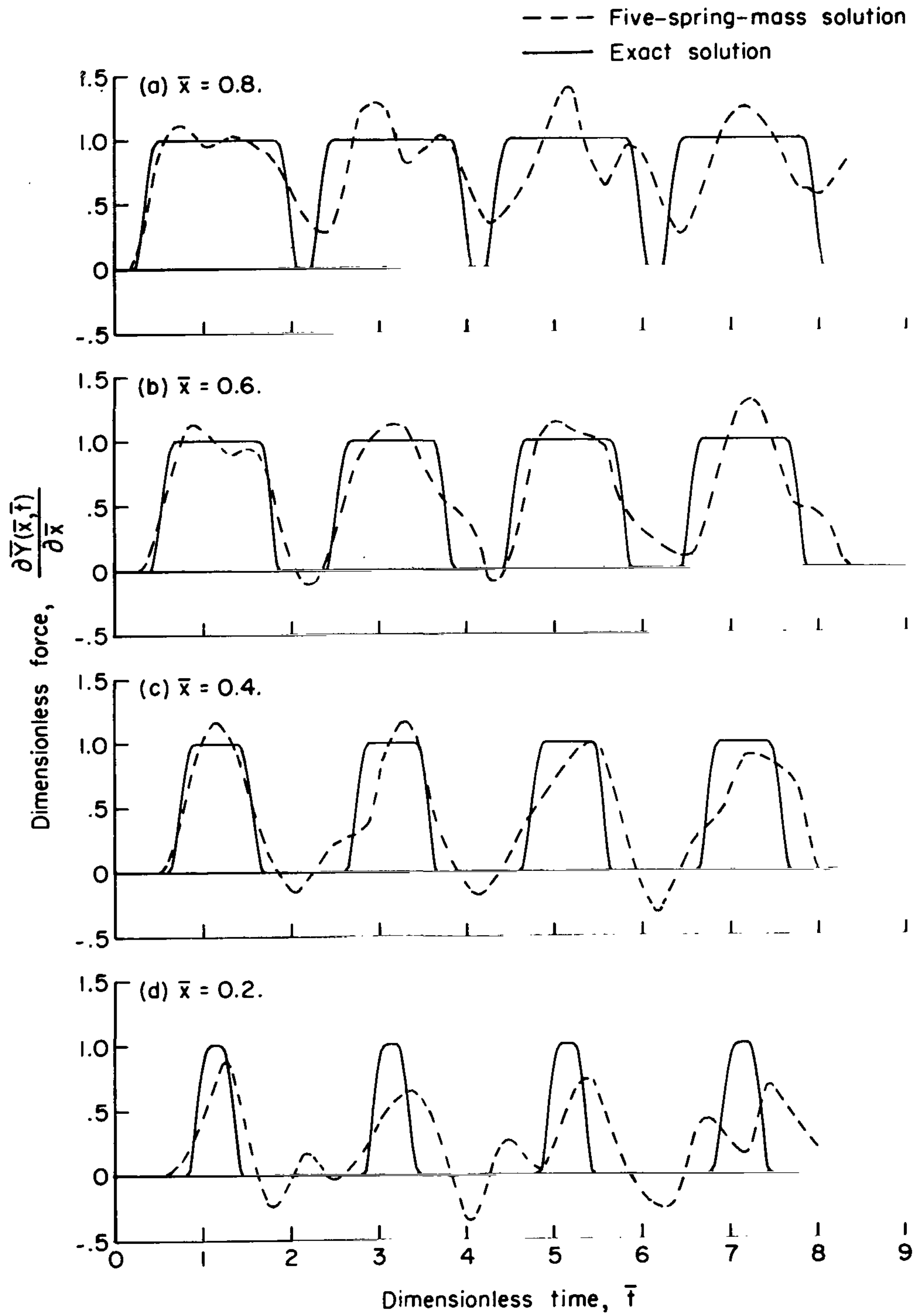



Figure 10.- Comparison between force $\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = p(\bar{t})$. (See fig. 2 for sketch of $p(\bar{t})$.)

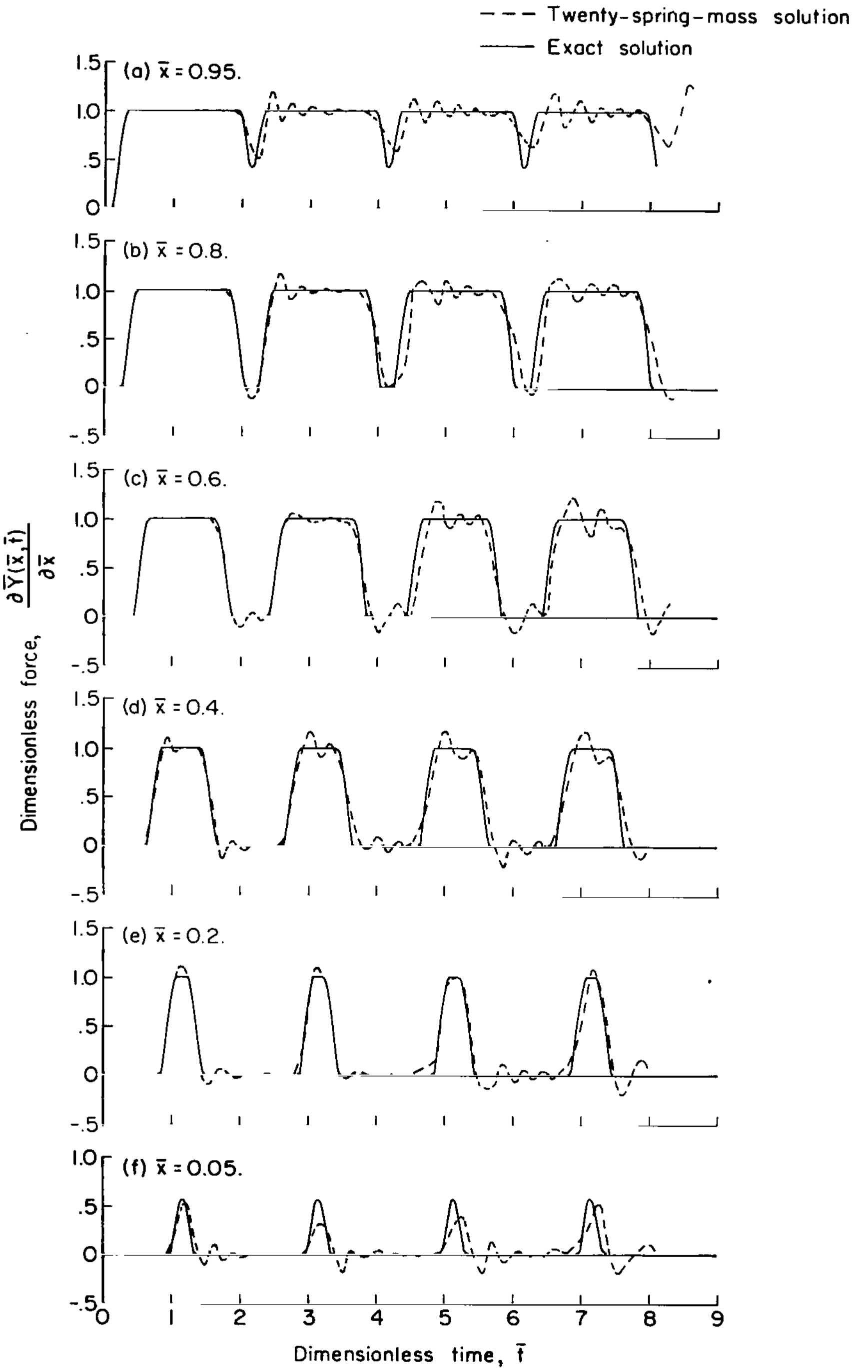



Figure 11.- Comparison between force $\frac{\partial \bar{Y}(\bar{x},\bar{t})}{\partial \bar{x}}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = p(\bar{t})$. (See fig. 2 for sketch of $p(\bar{t})$.)

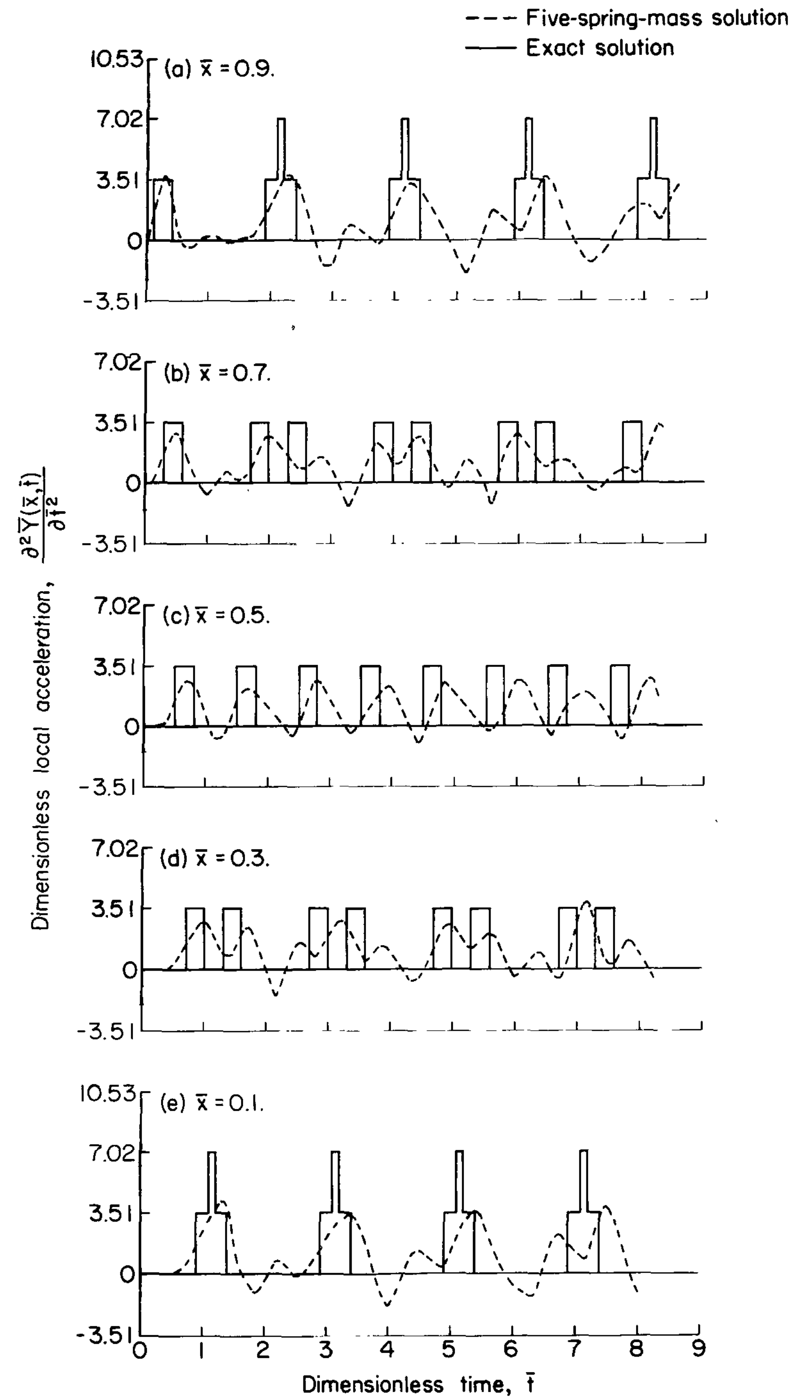



Figure 12.- Comparison between local acceleration $\frac{\partial^2 \bar{Y}(\bar{x},\bar{t})}{\partial \bar{t}^2}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = r(\bar{t})$. (See fig. 2 for sketch of $r(\bar{t})$.)

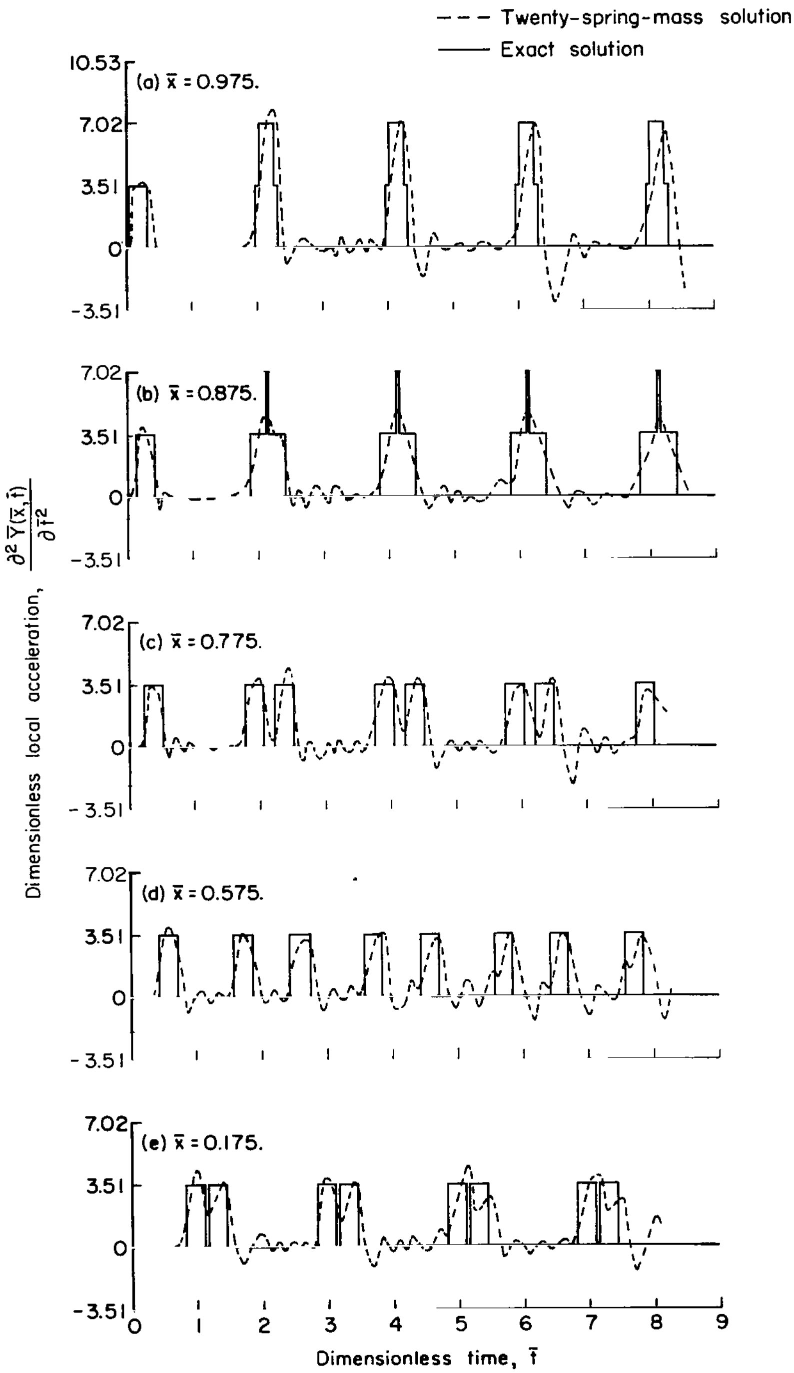



Figure 13.- Comparison between local acceleration $\frac{\partial^2 \overline{Y}(\overline{x},\overline{t})}{\partial \overline{t}^2}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\overline{F}(\overline{t}) = r(\overline{t})$. (See fig. 2 for sketch of $r(\overline{t})$.)

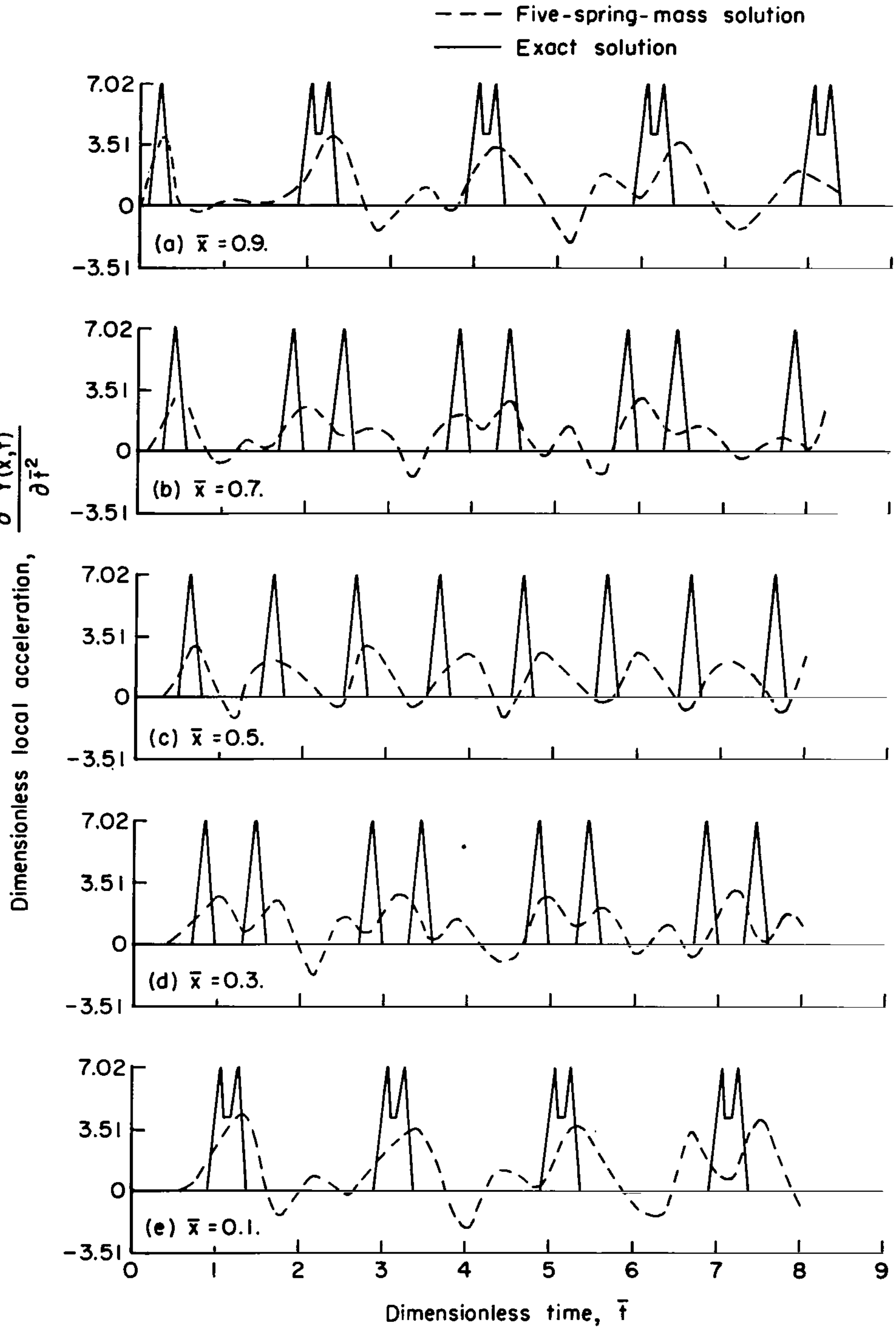



Figure 14.- Comparison between local acceleration $\frac{\partial^2 \bar{Y}(\bar{x},\bar{t})}{\partial \bar{t}^2}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = p(\bar{t})$. (See fig. 2 for sketch of $p(\bar{t})$.)

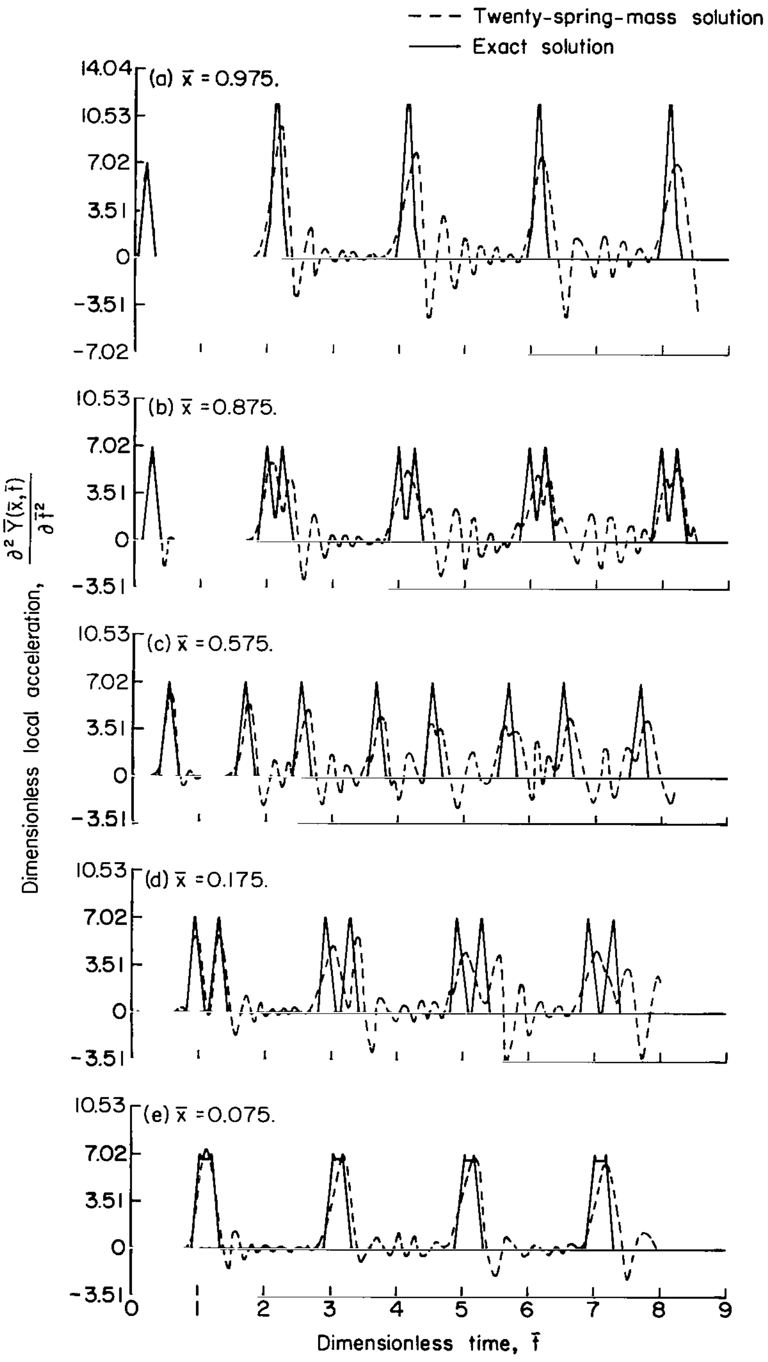



Figure 15.- Comparison between local acceleration $\frac{\partial^2 \bar{Y}(\bar{x},\bar{t})}{\partial \bar{t}^2}$ obtained by an exact and an approximate analysis for uniform bar with applied force $\bar{F}(\bar{t}) = p(\bar{t})$. (See fig. 2 for sketch of $p(\bar{t})$.)

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